AIAA JOURNAL
Vol. 43, No. 2, February 2005

Direct Least-Squares Formulation of a Stiffness Adjustment Method

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A direct least-squares formulation of the stiffness matrix adjustment method known as the KMA method (Kabe, A. M., "Stiffness Matrix Adjustment Using Mode Data," *AIAA Journal*, Vol. 23, No. 9, 1985, pp. 1431–1436) is presented. The KMA method belongs to a class of procedures that refine dynamic models using test-measured modes and structural connectivity. Viewed as a constrained minimization problem, most of these methods have applied Lagrange multiplier techniques in their development. It is shown that these two approaches result in linear systems of equations that are algebraically equivalent and therefore have identical solutions. By virtue of this equivalence, the direct least-squares formulation of the KMA method is shown to be equivalent to its original Lagrange multiplier formulation. The direct least-squares versions of other model refinement methods that preserve structural connectivity are also discussed. In this respect, this approach provides a rigorous framework for unifying these optimal update procedures and indicates how their solutions should compare. Numerical results are presented that illustrate the equivalence between the two formulations and also the conditions under which the various methods yield identical solutions. The results also indicate that because of improved numerical conditioning, the direct least-squares approach yields more accurate computational solutions and generally requires less computer storage than methods that were developed using Lagrange multipliers.

Nomenclature

$\boldsymbol{A}$ = $M \times N$ direct least-squares coefficient matrix
$\boldsymbol{B}$ = $n \times m$ eigendynamic constraint matrix, $[b_{ij}]$
$\boldsymbol{b}$ = M-dimensional constraint vector, $\{b_i\}$
$\boldsymbol{C}$ = $M \times N$ constraint coefficient matrix
$\boldsymbol{G}$ = $N \times N$ positive definite matrix, $[g_{ij}]$, that defines a cost metric
$\boldsymbol{H}$ = $M \times M$ Lagrange multiplier auxiliary matrix
$\boldsymbol{K}$ = $n \times n$ analytical stiffness matrix, $[k_{ij}]$
$\hat{\boldsymbol{K}}$ = $n \times n$ adjusted stiffness matrix, $[\hat{k}_{ij}]$
L = Lagrangian function
M = row dimension of $\boldsymbol{A}$
$\boldsymbol{M}$ = $n \times n$ mass matrix
m = number of modes
N = column dimension of $\boldsymbol{A}$
n = number of degrees of freedom
$\boldsymbol{P}$ = general matrix that premultiplies $\boldsymbol{A}$
R = rank of $\boldsymbol{A}$
$\boldsymbol{R}^N$ = N-dimensional real vector space
$\boldsymbol{U}$ = $M \times R$ left singular value decomposition factor of $\boldsymbol{A}$
$\boldsymbol{V}$ = $N \times R$ right singular value decomposition factor of $\boldsymbol{A}$
$\boldsymbol{W}$ = $n \times n$ symmetric weighting matrix, $[w_{ij}]$
$\boldsymbol{x}$ = N-dimensional optimization vector, $\{x_i\}$
$\boldsymbol{y}$ = N-dimensional direct least square vector
$\boldsymbol{Z}$ = squared reciprocal of nonzero elements of $\boldsymbol{W}$, $[z_{ij}]$
$\boldsymbol{\Gamma}$ = $\boldsymbol{P}^T\boldsymbol{P}$, matrix defining constraint metric
δ = relative norm difference between methods
δ_{ij} = Kronecker delta
ε = relative norm errors for KMA method
$\boldsymbol{\Theta}$ = $n \times n$ percent adjustment matrix, $[\theta_{ij}]$
$\boldsymbol{\Lambda}$ = $n \times m$ matrix of Lagrange multipliers
$\boldsymbol{\lambda}$ = M-dimensional vector of Lagrange multipliers
ν = uniformly distributed noise
$\boldsymbol{\rho}_i$ = transpose of ith row vector of $\boldsymbol{\Phi}$
$\boldsymbol{\Sigma}$ = $R \times R$ diagonal matrix of singular values of $\boldsymbol{A}$
$\boldsymbol{\Phi}$ = $n \times m$ modal matrix
$\boldsymbol{\varphi}_j$ = jth column of $\boldsymbol{\Phi}$
χ = cost function to be minimized
$\boldsymbol{\Psi}$ = $n \times m$ generalized momentum matrix
$\boldsymbol{\psi}_j$ = jth column of $\boldsymbol{\Psi}$
$\boldsymbol{\Omega}$ = $m \times m$ diagonal matrix of natural frequencies, ω_i, rad/s
$\| \|_2$ = Euclidean norm
$\| \|_F$ = Frobenius norm
$\| \|_G$ = norm induced by $\boldsymbol{G}$
$\odot$ = element-by-element matrix multiplication
$\rightarrow$ = operator that recasts a matrix row-wise to a vector
$\sim$ = accent denoting direct least-squares solution
$\dagger$ = superscript denoting generalized inverse

Introduction

DEVELOPMENT of large, complex spacecraft requires accurate analytical models for predicting dynamic responses, such as loads. Model verification by mode survey tests ensures that a mathematical model accurately represents a vehicle's dynamic characteristics. As part of the verification process, the analyst is faced with reconciling discrepancies between the analytical and test-measured modal parameters.

Over the years numerous investigators have contributed to the field of model refinement using test-measured modes. Mottershead and Friswell[1] present a good summary of work up to 1992. Since then, additional work has been accomplished, and Refs. 2–14 are representative of this work. In this paper we shall consider a method, first introduced by Kabe[15,16] as the KMA method, that updates the analytical stiffness matrix using mode data and structural connectivity information. Since then, Caesar and Peter,[17] Kammer,[18] and Smith and Beattie[19] have proposed the generalized linear least-squares (GLLS), projector matrix (PM), and multiple-secant Marwil–Toint (MSMT) methods, respectively. These are analogous methods that, under certain conditions, are mathematically equivalent to the KMA method. However, these methods do not guarantee that the resulting adjusted stiffness matrix is positive definite nor

Presented as Paper 2004-1531 at the AIAA/ASME/ASCE/AHS/ASC 45th Structures, Structural Dynamics, and Materials Conference, Palm Springs, CA, 19–22 April 2004; received 12 May 2004; revision received 30 July 2004; accepted for publication 3 August 2004.

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provide similar results when faced with inconsistent data. Smith and Beattie[20] and Beattie and Smith[21] provide an incisive review of these optimal update methods using inconsistent data and present two approaches to classifying these methods: 1) methods that seek to minimize the stiffness adjustments subject to the eigendynamic constraints and 2) methods that solve the eigendynamic constraints in a least-squares sense. The KMA, GLLS, and MSMT methods belong to the first view, whereas the PM method belongs to the second view.

A new and direct least-squares (DLS) formulation of the KMA method is presented. Originally it was viewed as a constrained minimization problem; Kabe used the Lagrange multiplier method (LMM) to define an auxiliary system of equations whose solution yielded the adjustment factors. The DLS formulation, on the other hand, adopts the second approach by solving the eigendynamic constraints directly in a least-squares sense. It is shown that with the appropriate metric on the space of admissible adjustment factors, the systems resulting from the LMM and DLS approaches are algebraically equivalent. The latter approach leads to a linear system of equations that is numerically better conditioned and requires fewer computational resources. We apply this algebraic result to mathematically compare previous stiffness-matrix updating procedures that preserve structural connectivity. It is shown that the DLS formulation of these optimal update methods provides a clear and rigorous basis for comparison. Numerical results of test problems are presented that illustrate the algebraic equivalence among the various methods and compare them.

Direct Least-Squares Formulation

We begin with the statement of a constrained minimization problem. Consider a metric on an N-dimensional vector space, $\boldsymbol{R}^N$, that is defined by a positive definite $N \times N$ matrix, $\boldsymbol{G}$. For $\boldsymbol{x} \in \boldsymbol{R}^N$, denote the resulting norm as

$$\|\boldsymbol{x}\|_G \equiv (\boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}x_i x_j\right)^{\frac{1}{2}} \tag{1}$$

We require that $\boldsymbol{x}$ satisfy the linear constraint

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{b} \tag{2}$$

where $\boldsymbol{C}$ is an $M \times N$ matrix and $\boldsymbol{b}$ is an M-dimensional constraint vector.

The constrained minimization problem is to find a solution $\boldsymbol{x}_0$ that satisfies Eq. (2) and minimizes the cost function

$$\chi(\boldsymbol{x}) = \|\boldsymbol{x}\|_G^2 \tag{3}$$

The solution is formally obtained by the method of Lagrange multipliers. Define the Lagrangian function

$$L = \tfrac{1}{2}\chi(\boldsymbol{x}) + \boldsymbol{\lambda}^T(\boldsymbol{b} - \boldsymbol{C}\boldsymbol{x}) \tag{4}$$

where $\boldsymbol{\lambda}$ is an M-dimensional vector of Lagrange mulitpliers. A necessary condition for a minimum is that it is a critical point of L. Equating the gradient of L to zero, we obtain the change of coordinates

$$\boldsymbol{x} = \boldsymbol{G}^{-1}\boldsymbol{C}^T\boldsymbol{\lambda} \tag{5}$$

Substituting Eq. (5) into Eq. (2) leads to the auxiliary system of linear equations associated with the Lagrange multiplier approach,

$$\boldsymbol{H}\boldsymbol{\lambda} = \boldsymbol{b} \tag{6}$$

where the $M \times M$ LMM auxiliary coefficient matrix is defined by

$$\boldsymbol{H} = \boldsymbol{C}\boldsymbol{G}^{-1}\boldsymbol{C}^T \tag{7}$$

Observe that $\boldsymbol{H}$ is a positive semidefinite matrix and has the same rank as $\boldsymbol{C}$. In general, $\boldsymbol{H}$ will be rank-deficient, and therefore a solution to Eq. (6) is given by the Moore–Penrose generalized inverse[22,23] of $\boldsymbol{H}$,

$$\boldsymbol{\lambda}_0 = \boldsymbol{H}^{\dagger}\boldsymbol{b} \tag{8}$$

Consequently, by Eq. (5), the constrained minimization problem has the solution

$$\boldsymbol{x}_0 = \boldsymbol{G}^{-1}\boldsymbol{C}^T\boldsymbol{H}^{\dagger}\boldsymbol{b} \tag{9}$$

We seek another approach to solving Eqs. (2) and (3) that uses the linear least-squares method, which is a standard approach for solving systems of linear equations that are rank deficient. The solution can be obtained by first computing the generalized inverse of the coefficient matrix via its singular value decomposition (SVD)[22–24] and then applying it to the constraint vector $\boldsymbol{b}$.

When $\boldsymbol{C}$ is viewed as a linear operator, the generalized inverse solution simultaneously satisfies minimization properties in both the range and domain of the operator. For overdetermined systems, the least-squares error is miminized by orthogonally projecting $\boldsymbol{b}$ onto the range of $\boldsymbol{C}$. This property ensures that the deviation from the constraint (2) is minimized. In cases where $\boldsymbol{C}$ is column rank deficient, the solution to the least-squares problem is not unique. However, of all the possible solutions, the generalized inverse solution is the unique one having the minimum Euclidean norm. This minimization property in the domain of $\boldsymbol{C}$, with the appropriate metric, allows us to obtain a solution with the minimum norm. Therefore, to incorporate the norm (1), introduce the change of coordinates

$$\boldsymbol{x} = \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{y} \tag{10}$$

Substituting Eq. (10) into Eq. (2) leads to the system of equations that defines the DLS formulation

$$\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \tag{11}$$

where

$$\boldsymbol{A} = \boldsymbol{C}\boldsymbol{G}^{-\frac{1}{2}} \tag{12}$$

Premultiplying Eq. (11) by $\boldsymbol{A}^{\dagger}$ and substituting into Eq. (10) yields the solution

$$\tilde{\boldsymbol{x}}_0 = \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{y}_0 = \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{A}^{\dagger}\boldsymbol{b} \tag{13}$$

We will use the tilde to denote DLS solutions. Observe that $\boldsymbol{y}_0$ is the unique least-squares solution to Eq. (11) with the smallest Euclidean norm. Since,

$$\|\boldsymbol{x}\|_G^2 = \boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{y} = \|\boldsymbol{y}\|_2^2 \tag{14}$$

we conclude that $\tilde{\boldsymbol{x}}_0$ is the unique least-squares solution to Eq. (2) that minimizes $\|\boldsymbol{x}\|_G$.

We claim that the LMM and DLS formulations are algebraically equivalent, and therefore yield the same solution. First note that the coordinate changes in Eqs. (5) and (10) associated with the LMM and DLS approaches, respectively, imply that

$$\boldsymbol{y} = \boldsymbol{A}^T\boldsymbol{\lambda} \tag{15}$$

Inspection of Eqs. (7) and (12) leads to the following factorization of the LMM auxiliary matrix:

$$\boldsymbol{H} = \boldsymbol{A}\boldsymbol{A}^T \tag{16}$$

Equations (15) and (16) show that the LMM auxiliary system (6) is a composition of a coordinate change from $\boldsymbol{y}$ to $\boldsymbol{\lambda}$ followed by the linear transformation $\boldsymbol{A}$. Figure 1 illustrates how these transformations commute. Identity (16) also implies that the LMM auxiliary coefficient matrices will always be positive semidefinite.

That the solutions from both formulations are equal follows from the SVD of $\boldsymbol{A}$,

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \tag{17}$$

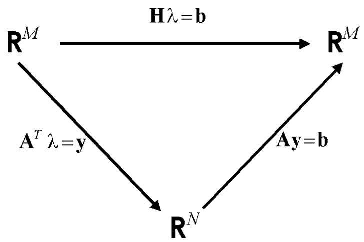

Fig. 1 Mapping diagram illustrating $H = AA^T$.

where $\boldsymbol{\Sigma}$ is the diagonal matrix of singular values. $\boldsymbol{U}$ and $\boldsymbol{V}$ are $M \times R$ and $N \times R$ orthonormal matrices, respectively. Also, the columns of $\boldsymbol{U}$ span the range of $\boldsymbol{C}$, whereas the columns of $\boldsymbol{V}$ span the orthogonal complement of the null space of $\boldsymbol{A}$. Substituting Eq. (17) into Eq. (16) yields the SVD of $\boldsymbol{H}$,

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T \tag{18}$$

Therefore, $\boldsymbol{H}^\dagger$ is obtained by inverting $\boldsymbol{\Sigma}^2$ in the earlier expression. Substituting this and Eq. (12) into Eq. (9) gives

$$\boldsymbol{x}_0 = \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{\Sigma}^{-2}\boldsymbol{U}^T\boldsymbol{b} \tag{19}$$

Equations (13) and (19), together with $\boldsymbol{A}^\dagger = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^T$, show that

$$\boldsymbol{x}_0 = \boldsymbol{G}^{-\frac{1}{2}}(\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{U}^T)(\boldsymbol{U}\boldsymbol{\Sigma}^{-2}\boldsymbol{U}^T)\boldsymbol{b} = \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{A}^\dagger\boldsymbol{b} = \tilde{\boldsymbol{x}}_0 \tag{20}$$

A few remarks are in order. Observe that a different metric $\boldsymbol{G}$ would yield different solutions only when $\boldsymbol{A}$ was column-rank-deficient, for example, when Eq. (11) was underdetermined. When the system is overdetermined, the projection of $\boldsymbol{b}$ onto the range of $\boldsymbol{A}$ determines a unique solution $\boldsymbol{x}$ regardless of the metric $\boldsymbol{G}$. However, premultiplying Eq. (11) by a matrix $\boldsymbol{P}$ essentially defines a new "constraint metric" on the range of $\boldsymbol{A}$ that modifies how $\boldsymbol{b}$ is projected. This leads to the generalization of the DLS method

$$\boldsymbol{PAy} = \boldsymbol{Pb} \tag{21}$$

whose solution minimizes the squared norm of the constraint residual,

$$\begin{aligned}\|\boldsymbol{PAy} - \boldsymbol{Pb}\|_2^2 &= (\boldsymbol{Ay} - \boldsymbol{b})^T[\boldsymbol{P}^T\boldsymbol{P}](\boldsymbol{Ay} - \boldsymbol{b}) \\ &= \|\boldsymbol{Ay} - \boldsymbol{b}\|_\Gamma^2, \qquad \boldsymbol{\Gamma} = \boldsymbol{P}^T\boldsymbol{P}\end{aligned} \tag{22}$$

In cases where the constraints cannot be satisfied, for example in overdetermined system with inconsistent data, the solutions to Eqs. (11) and (21) will generally differ. For overdetermined systems with full column rank, letting $\boldsymbol{P} = \boldsymbol{A}^T$ can provide significant reduction in memory requirements, especially if M is much larger than N. For this case, a straightforward SVD argument shows that the least-squares solutions of Eqs. (11) and (21) are identical.

Application to the KMA Method

We begin with a brief review of the KMA method and the auxiliary equations that result from Kabe's original approach, which used Lagrange multipliers. Let $\boldsymbol{K}$ denote an $n \times n$ analytical stiffness matrix that is to be adjusted. Then structural connectivity is maintained if the sparsity pattern of $\boldsymbol{K}$ is preserved. This requirement naturally leads to an adjusted stiffness matrix, $\hat{\boldsymbol{K}}$, defined by an element-by-element multiplicative adjustment matrix of the form

$$\hat{\boldsymbol{K}} = \boldsymbol{K} + \boldsymbol{K} \odot \boldsymbol{\Theta} \tag{23}$$

where $\boldsymbol{\Theta}$ is an $n \times n$ matrix that represents the percentage change in the stiffness matrix and has the same sparsity pattern as $\boldsymbol{K}$. To avoid unrealistic changes in the stiffness coefficients, the cost function, defined by a weighted Frobenius norm squared, is to be minimized:

$$\chi_{\text{KMA}} = \sum_{i,j=1}^{n} w_{ij}^2\theta_{ij}^2 \tag{24}$$

where $w_{ij} = 0$ if $k_{ij} = 0$.

To improve the correlation with the test-measured modes, Kabe imposed the following eigendynamic and symmetry constraints:

$$(\boldsymbol{K} \odot \boldsymbol{\Theta})\boldsymbol{\Phi} = \boldsymbol{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2 - \boldsymbol{K}\boldsymbol{\Phi} \equiv \boldsymbol{B} \tag{25}$$

$$\boldsymbol{\Theta}^T = \boldsymbol{\Theta} \tag{26}$$

where $\boldsymbol{M}$ is the $n \times n$ mass matrix, $\boldsymbol{\Phi}$ is the $n \times m$ matrix of mode shapes, and $\boldsymbol{\Omega}^2$ is the corresponding diagonal matrix formed by the square of the circular frequencies. The constraints are incorporated into the minimization of χ_{KMA} by the method of Lagrange multipliers and lead to the matrix analog of Eq. (5),

$$\boldsymbol{\Theta} = (\boldsymbol{Z} \odot \boldsymbol{K}) \odot (\boldsymbol{\Lambda}\boldsymbol{\Phi}^T + \boldsymbol{\Phi}\boldsymbol{\Lambda}^T) \tag{27}$$

where $\boldsymbol{\Lambda}$ is the $n \times m$ matrix of Lagrange multipliers and $\boldsymbol{Z} = [z_{ij}]$ is defined by

$$z_{ij} = \begin{cases} 0, & w_{ij} = 0 \\ w_{ij}^{-2}, & w_{ij} \neq 0 \end{cases} \tag{28}$$

Premultiplying Eq. (27) elementwise by $\boldsymbol{K}$ and substituting into Eq. (25) yields

$$[(\boldsymbol{Z} \odot \boldsymbol{K} \odot \boldsymbol{K}) \odot (\boldsymbol{\Lambda}\boldsymbol{\Phi}^T + \boldsymbol{\Phi}\boldsymbol{\Lambda}^T)]\boldsymbol{\Phi} = \boldsymbol{B} \tag{29}$$

Equation (29) is the matrix analog of the auxiliary system (6) in standard form. To facilitate the conversion to standard form, introduce the operator, $\rightarrow$, that casts an $n \times m$ matrix, rowwise, to an $nm \times 1$ vector via

$$\boldsymbol{B} \rightarrow \boldsymbol{b} = \{b_{11} \cdots b_{1m} \cdots b_{n1} \cdots b_{nm}\}^T \tag{30}$$

Likewise, casting $\boldsymbol{\Lambda} \rightarrow \boldsymbol{\lambda}$, we can express Eq. (29) in standard form (6), where the entries of the auxiliary matrix, $\boldsymbol{H}_{\text{KMA}}$, are given by Kabe[15,16] and has the same block sparsity pattern as $\boldsymbol{K}$. The derivation of Eq. (6) in the context of the KMA method could also have been accomplished by first casting Eqs. (25) and (26) into standard form using only the nonzero elements θ_{ij} corresponding to the upper triangular part of $\boldsymbol{\Theta}$, and then applying the Lagrange multiplier technique. This was the approach taken by Caesar and Peter[17] in developing the GLLS method. In general, both approaches require that an auxiliary $nm \times nm$ system of linear equations be solved in a least-squares sense.

The DLS formulation of the KMA method, like the GLLS method, begins by defining as its variables the percentage changes associated with the nonzero elements in the upper triangular part of the stiffness matrix. To reference the indices of these nonzero stiffness coefficients, denote the row-wise ordered set of their index pairs by

$$(i_p, j_p) : j_p \geq i_p \quad \text{and} \quad k_{i_p j_p} \neq 0, \qquad p = 1, \ldots, N \tag{31}$$

Represent the variables by the $N \times 1$ vector

$$\boldsymbol{x} = \left\{\theta_{i_1 j_1} \cdots \theta_{i_p j_p} \cdots \theta_{i_N j_N}\right\}^T \tag{32}$$

Then constraints (25) and (26) can be recast as Eq. (2) with $\boldsymbol{b}$ defined by the expression (30) and $M = nm$. The resulting $\boldsymbol{C}$ matrix is sparse and has its pth column equal to zero except for the i_pth and j_pth blocks equal to $k_{i_p j_p}\boldsymbol{\rho}_{j_p}$ and $k_{j_p i_p}\boldsymbol{\rho}_{i_p}$, respectively, where $\boldsymbol{\rho}_{i_p}$ is the transpose of the i_pth row vector of $\boldsymbol{\Phi}$.

The matrix that induces the norm associated with χ_{KMA} is given by $\boldsymbol{G}_{\text{KMA}} = [g_{rp;\text{KMA}}]$, where

$$g_{rp;\text{KMA}} = \begin{cases} 0, & \text{if} \quad r \neq p \\ w_{i_p i_p}^2, & \text{if} \quad r = p, i_p = j_p \\ 2w_{i_p j_p}^2, & \text{if} \quad r = p, i_p \neq j_p \end{cases} \tag{33}$$

Observe that the factor of two accounts for the off-diagonal coefficients appearing twice. Applying Eqs. (10–12), the DLS formulation of the KMA method is defined by

$$\boldsymbol{A}_{\text{KMA}}\,\boldsymbol{y} = \boldsymbol{b} \tag{34}$$

where

$$A_{\mathrm{KMA}} = CG_{\mathrm{KMA}}^{-\frac{1}{2}}, \qquad x = G_{\mathrm{KMA}}^{-\frac{1}{2}} y \tag{35}$$

By virtue of the factorization shown in the preceding section,

$$H_{\mathrm{KMA}} = A_{\mathrm{KMA}} A_{\mathrm{KMA}}^T \tag{36}$$

We add that the DLS method is equivalent to the LMM method even for test-measured modes that are not consistent.

Computationally, the DLS method should be superior to the LMM method. A heuristic counting argument shows that the number of modes needed to completely determine the N adjustment factors, $\theta_{i_p j_p}$ must satisfy

$$nm - m(m-1)/2 \geq N \tag{37}$$

Use of standard linear algebra software packages requires in-core storage of these coefficient matrices in their entirety. Therefore, the $nm \times N$ DLS system (34) requires less computer storage than the $nm \times nm$ LMM auxiliary system of equations. When A_{KMA} is of full column rank and $nm \gg N$, the generalized DLS with $P = A_{\mathrm{KMA}}^T$ further reduces the amount of internal memory needed. The use of sparse matrix solvers is currently being investigated, and we anticipate a significant reduction in computer memory requirements. We also expect the DLS system of equations to be numerically better conditioned. This is because the LMM auxiliary coefficient matrix consists of the stiffness terms squared, which vary over several orders of magnitude, and therefore would increase finite-precision errors.

Comparison to Other Methods

The DLS formulation provides a way to theoretically compare and unify optimal update methods that preserve structural connectivity. For the KMA, GLLS, and MSMT methods, the basis for comparison is the matrix G that is associated with the cost function that measures the percentage change in stiffness terms. Implicitly, these methods use the Euclidean metric on the range of A to measure how well the solution satisfies the eigendynamic constraints. Selection of a different constraint metric can produce dissimilar results when the data are inconsistent with the eigendynamic constraints. This is the main point of departure between the PM method and the other three methods mentioned previously.

The MSMT method, like the KMA method, uses the Lagrange multiplier technique to minimize the cost function

$$\chi_{\mathrm{MSMT}} = \sum_{i,j=1}^{n} \frac{(\hat{k}_{ij} - k_{ij})^2}{k_{ii}k_{jj}} = \sum_{i,j=1}^{n} \left(\frac{|k_{ij}|}{\sqrt{k_{ii}k_{jj}}} \right)^2 \theta_{ij}^2 \tag{38}$$

subject to constraints (25) and (26). Hence, the DLS formulation of the MSMT method becomes

$$A_{\mathrm{MSMT}} y = b, \qquad A_{\mathrm{MSMT}} = CG_{\mathrm{MSMT}}^{-\frac{1}{2}} \tag{39}$$

where the matrix $G_{\mathrm{MSMT}} = [g_{rp;\mathrm{MSMT}}]$ associated with the cost metric is defined by

$$g_{rp;\mathrm{MSMT}} = \begin{cases} 0, & \text{if} \quad r \neq p \\ 1, & \text{if} \quad r = p, i_p = j_p \\ 2k_{i_p j_p}^2 \left(k_{i_p i_p} k_{j_p j_p} \right)^{-1}, & \text{if} \quad r = p, i_p \neq j_p \end{cases} \tag{40}$$

Rederiving the MSMT auxiliary matrix in terms of θ_{ij}, we find that Eq. (16) also holds:

$$A_{\mathrm{MSMT}} A_{\mathrm{MSMT}}^T = F^T (I + \Pi) F \equiv H_{\mathrm{MSMT}} \tag{41}$$

where F and Π are the block diagonal and reordering matrices, respectively.[19] We add that Smith and Beattie's original numerical implementation solves the sparse auxiliary system of equations using a conjugate gradient method and hence requires less memory. However, the authors have noted convergence problems when faced with inconsistent data due to the positive semidefiniteness of H_{MSMT}.

The GLLS method starts by recasting the eigendynamic constraints into standard form (2) using only the upper triangular nonzero adjustment factors. The cost function to be minimized is defined by

$$\chi_{\mathrm{GLLS}} = \sum_{p=1}^{N} W_{p;\mathrm{GLLS}} \left(\hat{k}_{i_p j_p} - k_{i_p j_p} \right)^2 = \sum_{i,j=1}^{n} \left(k_{i_p j_p} \sqrt{W_{p;\mathrm{GLLS}}} \right)^2 \theta_{i_p j_p}^2 \tag{42}$$

The GLLS method proceeds to minimize χ_{GLLS} using Lagrange multipliers, and therefore requires the solution of an $nm \times nm$ auxiliary system of equations. The DLS coefficient matrix for the GLLS method is therefore given by

$$A_{\mathrm{GLLS}} = CG_{\mathrm{GLLS}}^{-\frac{1}{2}}, \; g_{rp;\mathrm{GLLS}} = \begin{cases} 0, & \text{if} \quad r \neq p \\ k_{i_p j_p}^2 W_{p;\mathrm{GLLS}}, & \text{if} \quad r = p \end{cases} \tag{43}$$

Caesar applied the GLLS method to Kabe's test problem,[15] using weighting factors equal to $k_{i_p j_p}^{-2}$, in order to minimize the percentange change. For this selection of weights, G_{GLLS} equals the identity matrix.

The PM method seeks to determine an updated stiffness matrix whose projection is defined by the test modes. The projection is based upon the fact that mass orthonormalized modes and their generalized momentums ($\psi_j = M\varphi_j$) form a dual basis

$$\psi_i^T \varphi_j = \delta_{ij} \tag{44}$$

and the resolution of the identity

$$\sum_{j=1}^{n} \varphi_j \psi_j^T = I \tag{45}$$

Given m test modes, Kammer defined the projection matrix

$$U_m = \sum_{j=1}^{m} \varphi_j \psi_j^T = \Phi \Psi^T, \qquad \Psi = M\Phi \tag{46}$$

and sought an updated stiffness matrix such that

$$\hat{K} U_m = K_m, \qquad K_m = \Psi \Omega^2 \Psi^T \tag{47}$$

which is equivalent to the eigendynamic constraints post-multiplied by Ψ^T. The solution of Eq. (46) is then obtained by recasting to standard form using only the nonzero upper triangular coefficients $\hat{k}_{i_p j_p}$ and equating the left side to the upper triangular elements of K_m. Incorporating the PM's ellipsoidal norm to minimize changes in a relative sense and expressing $\hat{k}_{i_p j_p}$ in terms of the percentage changes $\theta_{i_p j_p}$ lead to the generalized DLS formulation

$$P_{\mathrm{PM}} A_{\mathrm{PM}} y = P_{\mathrm{PM}} b \tag{48}$$

where P_{PM} is the $n(n+1)/2 \times nm$ block diagonal matrix

$$P_{\mathrm{PM}} = \begin{bmatrix} \Psi_1 & & & \\ & \Psi_2 & & \\ & & \ddots & \\ & & & \Psi_n \end{bmatrix}$$

$$\Psi_i = \begin{bmatrix} \psi_{i,1} & \psi_{i,2} & \cdots & \psi_{i,m} \\ \psi_{i+1,1} & \psi_{i+1,2} & \cdots & \psi_{i+1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n,1} & \psi_{n,2} & \cdots & \psi_{n,m} \end{bmatrix} \tag{49}$$

$$A_{\mathrm{PM}} = CG_{\mathrm{PM}}^{-\frac{1}{2}}, \qquad g_{rp;\mathrm{PM}} = \begin{cases} 0, & \text{if} \quad r \neq p \\ k_{i_p j_p}^2 W_{p;\mathrm{PM}}, & \text{if} \quad r = p \end{cases} \tag{50}$$

Table 1 Conditions for identical solutions

	Consistent		Inconsistent	
Method	UD[a]	OD[b]	UD	OD
KMA		√		√
MSMT		√		√
GLLS	√	√	√	√
PM	√	√	√	

[a]Underdetermined. [b]Overdetermined.

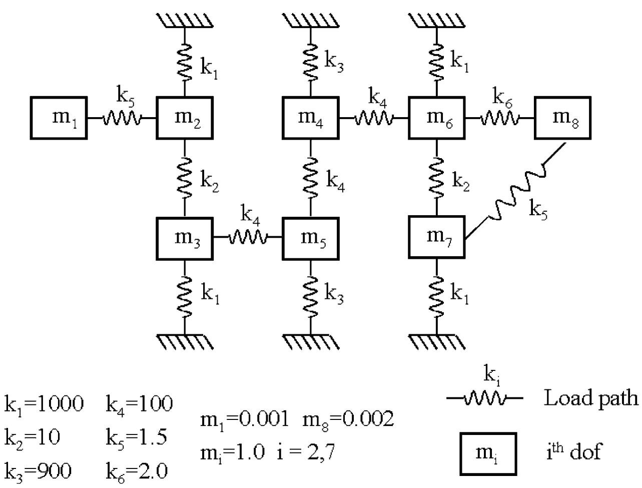

Fig. 2 Kabe's analytical test structure.

Defining the weights $W_{p;\mathrm{PM}} = k_{i_p j_p}^{-2}$ as suggested by Kammer leads to $\boldsymbol{G}_{\mathrm{PM}} = \boldsymbol{I}$. From Eq. (22), the least-squares solution to Eq. (48) minimizes the constraint residual

$$\|\boldsymbol{A}_{\mathrm{PM}}\boldsymbol{y} - \boldsymbol{b}\|_{\Gamma_{\mathrm{PM}}}^2, \qquad \Gamma_{\mathrm{PM}} = \boldsymbol{P}_{\mathrm{PM}}^T \boldsymbol{P}_{\mathrm{PM}} \tag{51}$$

We add that the PM method also provides a means of modifying the above metric by applying a diagonal weighting matrix that premultiplies $\boldsymbol{P}_{\mathrm{PM}}$.

Based on our earlier remarks, we summarize in Table 1 the conditions under which these optimal update methods yield identical solutions.

Numerical Results

This section provides numerical evidence that the LMM and DLS formulations of the KMA method are equivalent. We also present results that illustrate the conditions under which the methods yield the same solutions as listed in Table 1. The eight DOF analytical test structure shown in Fig. 2 will be considered.

The exact and corrupted analytical stiffness matrices were taken from Kabe[15] and are defined by Eqs. (52) and (53), respectively. Note that the number of nonzero upper triangular stiffness coefficients, N, equals 16. Ordering the θ_{ij} according to expression (31) and recasting row-wise define the percent adjustment vector and constraint matrix in Eqs. (54) and (55), respectively:

$$\boldsymbol{K}_E = \begin{bmatrix} 1.5 & -1.5 & & & & & & \\ -1.5 & 1011.5 & -10 & & & & & \\ & -10 & 1110 & & -100 & & & \\ & & & 1100 & -100 & -100 & & \\ & & -100 & -100 & 1100 & & & \\ & & & -100 & & 1112 & -10 & -2.0 \\ & & & & & -10 & 1011.5 & -1.5 \\ & & & & & -2.0 & -1.5 & 3.5 \end{bmatrix} \tag{52}$$

Table 2 Errors from the KMA method using consistent modes

m	ε_1	ε_2	ε_3	ε_4
1	4.27*E*–14	8.57*E*–01	8.57*E*–01	1.22*E*–16
2	6.46*E*–08	4.03*E*–01	4.03*E*–01	2.29*E*–16
3	1.01*E*–07	1.01*E*–07	4.84*E*–13	1.54*E*–16
4	1.98*E*–09	1.98*E*–09	5.35*E*–13	1.43*E*–16
5	5.49*E*–09	5.49*E*–09	2.44*E*–13	1.54*E*–16
6	1.18*E*–09	1.18*E*–09	9.67*E*–13	1.60*E*–16
7	6.11*E*–11	6.11*E*–11	1.82*E*–14	1.60*E*–16
8	7.93*E*–14	7.95*E*–14	8.78*E*–15	1.60*E*–16

$$\boldsymbol{K} = \begin{bmatrix} 2 & -2 & & & & & & \\ -2 & 1512 & -10 & & & & & \\ & -10 & 1710 & & -200 & & & \\ & & & 850 & -200 & -200 & & \\ & & -200 & -200 & 850 & & & \\ & & & -200 & & 1714 & -10 & -4 \\ & & & & & -10 & 1512 & -2 \\ & & & & & -4 & -2 & 6 \end{bmatrix} \tag{53}$$

$$\boldsymbol{x} = \{\theta_{11}\ \theta_{12}\ \theta_{22}\ \theta_{23}\ \theta_{33}\ \theta_{35}\ \theta_{44}\ \theta_{45}\ \theta_{46}\ \theta_{55}\ \theta_{66}\ \theta_{67}\ \theta_{68}\ \theta_{77}\ \theta_{78}\ \theta_{88}\}^T \tag{54}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_{11} & \boldsymbol{c}_{12} & & & & & & & & & & & & & & \\ & \boldsymbol{c}_{21} & \boldsymbol{c}_{22} & \boldsymbol{c}_{23} & & & & & & & & & & & & \\ & & & \boldsymbol{c}_{32} & \boldsymbol{c}_{33} & \boldsymbol{c}_{35} & & & & & & & & & & \\ & & & & & & \boldsymbol{c}_{44} & \boldsymbol{c}_{45} & \boldsymbol{c}_{46} & & & & & & & \\ & & & & & \boldsymbol{c}_{53} & & \boldsymbol{c}_{54} & & \boldsymbol{c}_{55} & & & & & & \\ & & & & & & & & \boldsymbol{c}_{64} & & \boldsymbol{c}_{66} & \boldsymbol{c}_{67} & \boldsymbol{c}_{68} & & & \\ & & & & & & & & & & & \boldsymbol{c}_{76} & & \boldsymbol{c}_{77} & \boldsymbol{c}_{78} & \\ & & & & & & & & & & & & \boldsymbol{c}_{86} & & \boldsymbol{c}_{87} & \boldsymbol{c}_{88} \end{bmatrix}$$

$$\boldsymbol{c}_{ij} = k_{ij}\boldsymbol{\rho}_j \tag{55}$$

The LMM and DLS formulations of the KMA method were applied using one through eight modes yielding solutions $\boldsymbol{\Theta}_{\mathrm{KMA}}$ and $\tilde{\boldsymbol{\Theta}}_{\mathrm{KMA}}$, respectively, for each value of m. Let $\boldsymbol{\Theta}_E$ denote the exact percentage change needed to adjust $\boldsymbol{K}$ correctly. Using consistent modes, Table 2 lists the relative errors as defined by

$$\varepsilon_1 = \frac{\|\tilde{\boldsymbol{\Theta}}_{\mathrm{KMA}} - \boldsymbol{\Theta}_{\mathrm{KMA}}\|_F}{\|\boldsymbol{\Theta}_E\|_F}, \qquad \varepsilon_3 = \frac{\|\tilde{\boldsymbol{\Theta}}_{\mathrm{KMA}} - \boldsymbol{\Theta}_E\|_F}{\|\boldsymbol{\Theta}_E\|_F}$$

$$\varepsilon_2 = \frac{\|\boldsymbol{\Theta}_{\mathrm{KMA}} - \boldsymbol{\Theta}_E\|_F}{\|\boldsymbol{\Theta}_E\|_F}, \qquad \varepsilon_4 = \frac{\left\|\boldsymbol{A}_{\mathrm{KMA}}\boldsymbol{A}_{\mathrm{KMA}}^T - \boldsymbol{H}_{\mathrm{KMA}}\right\|_F}{\|\boldsymbol{H}_{\mathrm{KMA}}\|_F} \tag{56}$$

The ε_1 errors in the second column show that, to within numerical precision, the LMM and DLS formulations of the KMA method yield the same answers. The errors in the third and fourth columns indicate that the DLS solutions are more accurate than solutions obtained from the LMM formulation. This is because the DLS equations are numerically better conditioned. The last column verifies that the factorization identity (36) holds up to machine precision.

Further inspection of the singular values of $\boldsymbol{A}_{\mathrm{KMA}}$ shows that Eq. (34) is underdetermined, with rank equal to 8 and 15 when one and two modes are used, respectively. This is consistent with inequality (37). For more than two modes, the DLS system is overdetermined, with $\boldsymbol{A}_{\mathrm{KMA}}$ having full column rank equal to 16. For these cases, the exact adjustment factors can be determined to within

Table 3 Errors from the KMA method using inconsistent modes with 10% error

m	ε_1	ε_2	ε_3	ε_4
1	1.90E–13	8.54E–01	8.54E–01	1.61E–16
2	2.87E–08	5.20E–01	5.20E–01	2.42E–16
3	2.58E–08	3.40E+00	3.40E+00	1.73E–16
4	9.76E–08	1.45E+00	1.45E+00	1.58E–16
5	6.60E–07	2.70E+00	2.70E+00	1.82E–16
6	1.39E–07	1.43E+00	1.43E+00	1.74E–16
7	4.16E–10	5.84E–01	5.84E–01	1.73E–16
8	7.04E–13	1.90E–01	1.90E–01	1.73E–16

Table 4 Relative differences among the methods using consistent modes

m	δ_1	δ_2	δ_3	δ_4
1	8.73E+00	8.60E–02	8.60E–02	1.49E–12
2	8.60E–01	2.93E–02	2.93E–02	1.93E–12
3	4.04E–13	1.91E–13	3.76E–10	3.76E–10
4	4.87E–13	4.21E–13	1.04E–11	1.04E–11
5	1.09E–13	2.36E–13	2.15E–12	2.10E–12
6	1.41E–12	1.03E–12	1.63E–12	1.46E–12
7	9.73E–14	7.09E–14	5.93E–12	6.00E–12
8	3.77E–15	5.68E–16	4.69E–15	4.90E–15

Table 5 Relative differences among the methods using inconsistent modes with 10% error

m	δ_1	δ_2	δ_3	δ_4
1	9.18E+00	8.84E–02	8.84E–02	2.69E–13
2	8.03E–01	2.91E–02	2.91E–02	9.00E–13
3	3.63E–13	6.35E–13	5.86E+00	5.86E+00
4	3.24E–13	5.66E–13	3.02E+00	3.02E+00
5	1.17E–12	7.09E–13	1.14E+00	1.14E+00
6	2.38E–13	9.19E–13	7.17E–01	7.17E–01
7	1.53E–13	1.38E–13	2.57E–01	2.57E–01
8	1.28E–14	1.23E–15	9.89E–15	8.98E–15

numerical accuracy. These observations are consistent with the errors listed in columns 3 and 4 of Table 2.

In practice, test-derived modes are subject to errors and are therefore inconsistent with the sparsity pattern of $\boldsymbol{K}$ and the eigendynamic constraints. To show that algebraic equivalence remains in force even with inconsistent data, the DLS and LMM formulations of the KMA method were applied to imperfect modes. These modes were obtained by multiplying each term of the analytical modes by $1 + \nu$, where ν was taken from a uniform distribution between ± 0.1. These corrupted modes were then orthonormalized with respect to the mass matrix using the procedure described by Targoff.[25]

The results are tabulated in Table 3. As noted by Kabe,[16] 10% errors in the test modes result in unacceptable adjustments in the stiffness coefficients. In spite of this, however, the second column shows that the DLS and LMM formulations yield identical solutions. Also, the errors listed in the fifth column show that factorization identity remains true. We add that similar results were obtained when the DLS and LMM formulations of the MSMT method were compared.

To numerically verify Table 1, which lists the conditions for the four methods to yield identical solutions, their DLS formulations were applied using exact and corrupted modes. Tables 4 and 5 list the relative differences in solutions as defined by Eq. (57) using exact modes and 10% corrupted modes, respectively:

$$\delta_1 = \frac{\|\tilde{\Theta}_{\mathrm{MSMT}} - \tilde{\Theta}_{\mathrm{KMA}}\|_F}{\|\Theta_E\|_F}, \qquad \delta_3 = \frac{\|\tilde{\Theta}_{\mathrm{PM}} - \tilde{\Theta}_{\mathrm{KMA}}\|_F}{\|\Theta_E\|_F}$$

$$\delta_2 = \frac{\|\tilde{\Theta}_{\mathrm{GLLS}} - \tilde{\Theta}_{\mathrm{KMA}}\|_F}{\|\Theta_E\|_F}, \qquad \delta_4 = \frac{\|\tilde{\Theta}_{\mathrm{PM}} - \tilde{\Theta}_{\mathrm{GLLS}}\|_F}{\|\Theta_E\|_F} \tag{57}$$

Table 4 shows that if consistent modes are used, the solutions from all four methods are identical when the system is overdetermined; that is, $m \geq 3$. On the other hand, when the system is underdetermined, there are nonunique solutions that satisfy the eigendynamic constraint. In this case, the metric $\boldsymbol{G}$ selects the solution with the minimum norm. Because the PM and the GLLS methods both use $\boldsymbol{G} = \boldsymbol{I}$, we see that they both yield the same solutions.

Table 5 supports the claims in Table 1 when inconsistent data are used. For the case where the system is overdetermined, the MSMT and GLLS methods give results identical to those of the KMA method, because they use the Euclidean metric to measure deviations from the eigendynamic constraints. The PM method, on the other hand, uses a different constraint metric defined by $\boldsymbol{P}_{\mathrm{PM}}^T \boldsymbol{P}_{\mathrm{PM}}$ and hence selects a different solution. Observe that if all modes are used in the example problem, the four methods yield the same solution. Also, as in Table 4, the solutions from PM and GLLS methods are identical when the system is underdetermined.

Some Practical Considerations

Updating large finite element models to match empirical modal data is a very difficult task. Because the number of parameters is extremely large, nonunique analytical models can be developed that will reproduce the available modal data. By increasing the number of constraints, such as preserving structural connectivity, one can improve the probability of obtaining an adjusted model closer to the actual system. However, this does assume that the structural connectivity of the original model is correct and that any geometric and element properties that result in zero terms in the stiffness matrix are also correct. As pointed out by Smith and Beattie,[19] certain truss geometries and stiffnesses can yield zero terms in the stiffness matrix. If these are to be included in the set of potential elements that can be updated, then the properties in the original matrix should be slightly changed to yield nonzero terms.

Another advantage of the KMA proceedure is the fact that the percentage change in stiffness coefficients is minimized. This seems reasonable, unless one can justify more confidence in some parts of the model than in other parts. If this is the case, weighing factors[16] can be included to minimize changes in those parts of the model felt to be more accurate.

As discussed in the original publication of the KMA approach, any results from model-updating procedures should be treated as indicators that need to be verified by a review of the drawings and the actual hardware before being implemented in the finite element model. This should remain a "manual" process because only by having the analysts in the loop can we have confidence that the physics of the problem is not being violated.

Finally, a primary purpose of this paper is to argue that some consensus has been built on the benefits of including structural connectivity as additional constraints in the model-adjustment process.[15–20] In addition, with the significant increase in computational power since the KMA procedure was originally introduced, and the work presented in this paper that allows an efficient direct least-squares solution of the KMA procedure, it is suggested that the time may be near when this approach could be used in production environments to develop indicators for adjusting analytical models.

Summary

A direct least-squares formulation for solving the constrained minimization problem in the KMA method was presented and shown to be algebraically equivalent to Kabe's original LMM formulation. The equivalence is a consequence of a general result that relates the systems of equations resulting from the Lagrange multiplier and least-squares approaches to solving a constrained minimization problem. This equivalence also extends to the MSMT and GLLS methods and, in a generalized sense, to the PM method. In this respect, the DLS formulation provides a rigorous framework for unifying these optimal update methods. Numerical results were presented that verified the equivalence between the LMM and DLS formulations and also the conditions under which the various methods yield identical solutions. The results also indicate that because of improved numerical conditioning, the DLS formulation yields more accurate computational solutions. Additionally, the DLS formulation requires less computer storage than methods that were derived using Lagrange multipliers.

References

[1]Mottershead, J. E., and Friswell, M. I., "Model Updating in Structural Dynamics: A Survey," *Journal of Sound and Vibration*, Vol. 167, No. 2, 1993, pp. 347–375.

[2]Alvin, K. F., and Park, K. C., "Extraction of Substructural Flexibility from Global Frequencies and Mode Shapes," *AIAA Journal*, Vol. 37, No. 11, 1999, pp. 1444–1451.

[3]Alvin, K. F., "Finite Element Model Update via Bayesian Estimation and Minimization of Dynamic Residuals," *AIAA Journal*, Vol. 35, No. 5, 1997, pp. 879–886.

[4]Cha, P. D., and Gu, W., "Model Updating Using an Incomplete Set of Experimental Modes," *Journal of Sound and Vibration*, Vol. 233, No. 4, 2000, pp. 587–600.

[5]Cha, P. D., "Correcting System Matrices Using the Orthogonality Conditions of Distinct Measured Modes," *AIAA Journal*, Vol. 38, No. 4, 2000, pp. 730–732.

[6]Denoyer, K. K., and Peterson, L. D., "Method for Structural Model Update Using Dynamically Measured Static Flexibility Matrices," *AIAA Journal*, Vol. 35, No. 2, 1997, pp. 362–368.

[7]Denoyer, K. K., and Peterson, L. D., "Model Update Using Modal Contribution to Static Flexibility Error," *AIAA Journal*, Vol. 35, No. 11, 1997, pp. 1739–1745.

[8]Denoyer, K. K., and Peterson, L. D., "Comparing Model Update Error Residuals and Effects on Model Predictive Accuracy," *AIAA Journal*, Vol. 36, No. 10, 1998, pp. 1894–1900.

[9]Friswell, M. I., Inman, D. J., and Pilkey, D. F., "Direct Updating of Damping and Stiffness Matrices," *AIAA Journal*, Vol. 36, No. 3, 1998, pp. 491–493.

[10]Kim, Hyoung M., and Bartkowicz, Theodore J., "Damage Detection and Health Monitoring of Large Space Structures, *Journal of Sound and Vibration*, Vol. 27, No. 6, 1993, pp. 12–17.

[11]Lollock, J. A., Cole, T. R., Ankrom, M. E., and Cooley, W. U., "A Method for Developing a High Fidelity Test-Corrected Dynamic Model," AIAA Paper 98-1893, 1998.

[12]Misawa, Masayoshi, "A Method to Find Accurate Mass and Stiffness Matrices in System Identification," AIAA Paper 2000-1745, 2000.

[13]Zimmerman, D. C., and Simmermacher, T., "Model Correlation Using Multiple Static Load and Vibration Tests," *AIAA Journal*, Vol. 33, No. 11, 1995, pp. 2182–2188.

[14]Zimmerman, D. C., Simmermacher, T., and Kaouk, M., "Model Correlation and System Health Monitoring Using Frequency Domain Measurements," AIAA Paper 95-1113, 1995.

[15]Kabe, A. M., "Stiffness Matrix Adjustment Using Mode Data," *AIAA Journal*, Vol. 23, No. 9, 1985, pp. 1431–1436.

[16]Kabe, A. M., "Constrained Adjustment Using of Analytical Stiffness Matrices," Society of Automotive Engineers, SAE Paper 851932, Oct. 1985.

[17]Caesar, B., and Peter, J., "Direct Update of Dynamic Mathematical Models from Modal Test Data," *AIAA Journal*, Vol. 25, No. 11, 1987, pp. 1494–1499.

[18]Kammer, D. C., "Optimum Approximation for Residual Stiffness in Linear System Identification," *AIAA Journal*, Vol. 26, No. 1, 1988, pp. 104–112.

[19]Smith, S. W., and Beattie, C. A., "Secant-Method Adjustment for Structural Models," *AIAA Journal*, Vol. 29, No. 1, 1991, pp. 119–126.

[20]Smith, S. W., and Beattie, C. A., "Optimal Identification Using Inconsistent Modal Data," AIAA Paper 91-0948, 1991.

[21]Beattie, C. A., and Smith, S. W., "Optimal Matrix Approximants in Structural Identification," *Journal of Optimization Theory and Applications*, Vol. 74, No. 1, 1992, pp. 23–56.

[22]Horn, R. A., and Johnson, C. R., *Matrix Analysis*, Cambridge Univ. Press, New York, 1985, pp. 427–454.

[23]Stewart, *Introduction to Matrix Computations*, Academic Press, New York, 1973, pp. 317–326.

[24]Golub, G. H., and Van Loan, C. F., *Matrix Computations*, Johns Hopkins Univ. Press, Baltimore, MD, 1983, pp. 136–188.

[25]Targoff, W. P., "Orthogonality Check and Correction of Measured Modes," *AIAA Journal*, Vol. 14, No. 2, 1976, pp. 164–167.

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